The questions below are modified from Section 3.3, P77 of Bartle.

1*. Let x(1) = 8 and x(n+1): = 2 + (x(n)/2). Show the sequence is decreasing and positive and hence (?) converges (to a finite limit). Find the limit.

2. Let f(x) : = 2 - 1/x for each positive x>1. Show that f(x) < x (Hint: look at some quadratic relation) and hence that the sequence x(n) defined in Q2 of Section 3.3 is decreasing to limit 1:
x(1) > 1 and x(n+1) = 2 - 1/x(n).

3. Let g(x) be defined by

$$g(x) = 1 + \sqrt{x-1} \quad \forall\ x \in [2, +\infty)$$

(Thus g is a 'self-map' mapping the domain of definition into itself). Show that g(x) is dominated by x and that the sequence x(n) defined by x(1) = any number in the interval and x(n+1) = g(x(n)) is decreasing. Find its limit.

4. Let $x_1 = 1$ and $x_{n+1} = \sqrt{2+x_n}$ $\forall n$. Then $(x_n)\uparrow$ and bounded by $1+\sqrt{2}$ (by M I). Find the limit if exists.

5. Let $p > 0$ and $y_1 = \sqrt{p}$, $y_{n+1} = \sqrt{p + y_n}$. Then $(y_n)\uparrow$ and bounded by $1+\sqrt{p}$. Find the limit if exists.

6. Let $a > 0$, $z_1 > 0$ and $z_{n+1} = \sqrt{a + z_n}$ $\forall n$. Then

$z_n \le M + \sqrt{a} := \max\{1, z_1\} + \sqrt{a}$ $\forall\ n \in \mathbb{N}$.

[Hint: Assume $z_n \le M + \sqrt{a}$. Then, as $M \le M^2$ & $1 \le M$,

$a + z_n \le a + M + \sqrt{a} \le (M + \sqrt{a})^2 = (a + M^2 + 2M\sqrt{a})$

So $z_{n+1} = \sqrt{a + z_n} \le M + \sqrt{a}$.]. Depending $z_1 \le z_2$ (or $z_1 > z_2$) one can show $(z_n)\uparrow$ (or $\downarrow$). So you can find your limit always.

7. $x_1 = a > 0$ and $x_{n+1} = x_n + \frac{1}{x_n}$ $\forall n$. Then $(x_n)\uparrow$, and hence $0 \ne x := \lim_n x_n \le +\infty$ (why). Should x be finite, we will have a contradiction by

passing to the limits in $x_{n+1} = x_n + \frac{1}{x_n}$ so

(x_n) must be unbounded & $\lim_n x_n = +\infty$.

Note. Unboundedness can also be proved in the following way: Since $x_{n+1}x_n = x_n^2 + 1$ and $(x_n)\uparrow$, one has $x_{n+1}^2 \geq x_n^2 + 1$ $\forall n$; hence (x_n^2) unbounded and so is (x_n).

8*. Let $\emptyset \neq A \subseteq \mathbb{R}$, bounded with $x := \sup A \in \mathbb{R}$. Show that $\exists$ a seq (x_n) in A such that $\lim_n x_n = x$. Moreover, if $x \notin A$ show that you can have your (x_n) satisfying additionally that $x_n < x_{n+1}$ $\forall n$.

9*. Let (a_n) be a bounded sequence, and

$$t_n = \inf\{a_m : m \geq n\} = \inf\{a_n, a_{n+1}, a_{n+2}, \cdots\}$$

$$s_n = \sup\{a_m : m \geq n\} = \sup\{a_n, a_{n+1}, a_{n+2}, \cdots\}.$$

Show that $(t_n), (s_n)$ are monotone and

$$\lim_n t_n = \sup\{t_n : n \in \mathbb{N}\} \leq \inf\{s_K : K \in \mathbb{N}\} = \lim_K s_K.$$

10*. Let $(a_n), (t_n), (s_n)$ be as in Q9. Show that (a_n) converges iff $\lim_n t_n = \lim_n s_n$.

$\lim_n t_n$ is usually denoted by $\liminf_n a_n$ (lower limit of (a_n))

$\lim s_n$ — — ——————— $\limsup_n a_n$ (upper limit of (a_n))